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Electrostatic and magnetostatic solutions in a Lorentz-violating electrodynamics model

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Abstract. We propose an effective Lorentz-violating electrodynamics model via the static de Sitter metric, which is deviated from the Minkowski metric by a minuscule amount depending on the cosmological constant. We obtain the electromagnetic field equations via the vierbein decomposition of the tensors. In addition, as an application of the electromagnetic field equations obtained, we derive the solutions of the electrostatic field and the magnetostatic field due to a point charge and a circle current, respectively, and discuss the implication of the effect of Lorentz violation in our electromagnetic theory.

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Lorentz invariance is one of the greatest discoveries in the history of physics and has been confirmed to ever greater precision. Most of the evidence comes from short distance tests. However, there are many signs that something strange may happen at a large distance (for example, dark energy), where the constraints on Lorentz symmetry are much weaker. It is reasonable that many researchers are interested in Lorentz violation (LV) from various points of views [1–5]. Researchers have pointed out that Lorentz invariance can be viewed as a low energy effective invariance. Remarkably, under suitable circumstances, some experimental information about quantum gravity can nonetheless be obtained. The point is that minuscule effects emerging from the underlying quantum gravity might be detected in sufficiently sensitive experiments. To be identified as definitive signals at the Planck scale, such effects would need to violate some established principles of low-energy physics. One promising class of potential effects is relativity violations, arising from breaking the Lorentz symmetry that lies at the heart of relativity. Recent proposals suggest LV effects may emerge from strings, loop quantum gravity, noncommutative field theories, or numerous other sources at the Planck scale [8]. On the other hand, recent observations, such as the luminosity observations of the farthest supernovas [9], show that our universe is accelerated expanding and probably asymptotically de Sitter with a positive cosmological constant Λ [10–12].

Among the developments on LV research is a systematic extension of the standard model of particle physics incorporating all possible LV in the renormalizable sector, called the standard model extension (SME), developed by Colladay and Kostelecký [6, 13]. This model has provided a framework for computing in effective field theory the observable consequences for many experiments and has led to much experimental work setting limits on the LV parameters in the Lagrangian [14]. The action of SME incorporates the standard model (SM) of particle physics, including gravitational couplings and a purely gravitational sector. The action of the effective theory is expected to contain the usual minimal gravitational coupling and the Einstein–Hilbert action among its terms. The photon sector of the QED extension in the SME framework is characterized by the LV coefficients CPT even k_F, CPT odd k_{AF} and k_A.

We limit our attention in the present work to the sector of classical Lorentz-violating electrodynamics, coupled to an arbitrary 4-current source. Due to the presence of dark energy or the nonzero positive cosmological constant, the spacetime without any matter is de Sitter rather than Minkowskian and so it is natural to substitute the Lorentz invariant low energy effective theory with its covariant formulation in de Sitter spacetime for a field theory in Minkowski spacetime. We set up the model as the electromagnetic field theory in de Sitter spacetime and take the vierbein or the local Lorentz frame formalism of the theory as its Minkowski spacetime limit. The Lorentz invariance is violated obviously in this way to the observer in Minkowski spacetime. We mean in this approach that the Lorentz symmetry is an approximate symmetry of the low energy effective theory. There are different types of metrics for de Sitter spacetime and it is well-known that quantum field theory in de Sitter spacetime equipped with the static metric is a finite temperature field theory in a pure field theory in the curved spacetime approach [15]. However, we investigate the Lorentz-violating electrodynamics in Minkowski

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spacetime limit in the present work and thus there is no finite temperature problem here. In this way, different choices of de Sitter metrics to formulate the Lorentz violating theory just follow different scenarios to violate Lorentz symmetry. In this paper, we propose a scenario that the substitute of Lorentz invariant electromagnetic theory is its formulation in de Sitter spacetime equipped with the static metric, which is obviously Lorentz-violating. We define all the observables in Minkowski spacetime as the corresponding vierbein decomposition components in de Sitter spacetime of corresponding physical quantities.

We borrow some ideas from SME here. However, our approach is a little different from the SME of Kostelecký et. al. In the SME approach, the LV part of the Minkowski spacetime limit of the electrodynamics formulation is a low energy effective Lagrangian consisting of CPT even k_F, CPT odd k_{AF} and k_A terms phenomenologically, which may originate from Riemann–Cartan gravitational coupling in the matter and gauge sectors. We stress the fact that the spacetime of our universe is asymptotic de Sitter and hence the low energy effective field theory should be one in de Sitter spacetime, which is deviated from field theory in Minkowski spacetime. The Lorentz violation arises naturally for the observer in the local Lorentz frame when he thinks he was in Minkowski spacetime. This means that we only investigate the Lorentz violation originating from dark energy or the cosmological constant in the present work.

First, we introduce the de Sitter space and its metric. de Sitter space can be regarded as a 4D hyperboloid S_R embedded in a 5D Minkowski space with $\eta_{AB} = diag(1,-1,-1,-1,-1)$,

$$
\begin{aligned}
&S_R : \eta_{AB}\xi^A\xi^B = -R^2 ,\\
&\mathrm{d}s^2 = \eta_{AB}\mathrm{d}\xi^A d\xi^B ,
\end{aligned} \tag{1}
$$

where $A, B = 0, ..., 4$. Clearly, (1) is invariant under de Sitter group $SO(1,4)$. The metric of this spacetime can be written as [16]

$$
\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}\xi^\mu\mathrm{d}\xi^\nu - \frac{(\eta_{\mu\nu}\xi^\mu\mathrm{d}\xi^\nu)^2}{1+K(\eta_{\mu\nu}\xi^\mu\xi^\nu)} , \tag{2}
$$

where $\mu,\nu = 0,...,3$, $K = \frac{1}{R^2} = \frac{\Lambda}{3}$,and Λ is the cosmological constant. This metric is invariant under two classes of simple transformations (see, for example, p. 387 of the book [16]): (i) $SO(1,3)$ transformations:

$$
\xi'^{\mu} = L^\mu{}_\nu \xi^\nu \tag{3}
$$

and (ii) "quasitranslations", with

$$
\xi'^{\mu} = \xi^\mu + a^\mu\left[(1-K\eta_{\rho\sigma}\xi^\rho\xi^\sigma)^{1/2} - bK\eta_{\rho\sigma}\xi^\rho a^\sigma\right] \tag{4}
$$

$$
b = \frac{1-(1-K\eta_{\rho\sigma}a^\rho a^\sigma)^{1/2}}{K\eta_{\rho\sigma}a^\rho a^\sigma} .
$$

In particular, these transformations take the origin $\xi^\mu = 0$ into any a^μ. For the metric given by (2), we can introduce coordinates in which the metric appears time-independent by

$$
\begin{aligned}
x^i &= \xi^i = x'^i \exp\left(K^{1/2}t'\right) ,\\
\xi^0 &= \frac{1}{\sqrt{K}}\left[\frac{K\mathbf{x}'^2}{2}\cosh\left(K^{1/2}t'\right)\right.\\
&\quad \left. + \left(1+\frac{K\mathbf{x}'^2}{2}\right)\sinh\left(K^{1/2}t'\right)\right] ,\\
t &= t' - \frac{1}{2K^{1/2}}\ln\left[1-K\mathbf{x}'^2\exp(2K^{1/2}t')\right] .
\end{aligned} \tag{5}
$$

Then (2) becomes

$$
\mathrm{d}s^2 = \left(1-K\mathbf{x}^2\right)\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2 - \frac{K\left(\mathbf{x}\cdot\mathrm{d}\mathbf{x}\right)^2}{1-K\mathbf{x}^2} . \tag{6}
$$

One can find that the spatial metric of the spacetime is just the metric of a 3D spherical surface (with radius R) in 4D Euclidean space. However, unlike the metric given by (2), this static de Sitter metric is obviously Lorentz violating. Noting that the transformation (4) and (5) leaving the metric (6) invariant, can also take the spatial origin $\mathbf{x} = \mathbf{0}$ into any $\mathbf{a}$ while keeping t unchanged and contain the spatial $SO(3)$ rotation. Choosing the spherical coordinate, we can rewrite the static metric (6) as follows:

$$
\mathrm{d}s^2 = \sigma\mathrm{d}t^2 - \frac{1}{\sigma}dr^2 - r^2\mathrm{d}\theta^2 - r^2\sin\theta^2\mathrm{d}\phi^2 . \tag{7}
$$

Thus we can define a local Lorentz frame with vierbeins ϑ^a_μ, $a = 0,1,2,3$, where

$$
\vartheta^a_\mu = \mathrm{diag}\left(\sqrt{\sigma}, \frac{1}{\sqrt{\sigma}}, r, r\sin\theta\right) . \tag{8}
$$

We now set up the LV electrodynamics by the vierbein formalism. A basic object in the formalism is the vierbeins ϑ^a_μ, which can be viewed as providing at each point on the spacetime manifold a link between the covariant components $T_{\lambda\mu\nu...}$ of a tensor field in a coordinate basis and the corresponding covariant components $T_{abc...}$ of the tensor field in a local Lorentz frame. The link is given by

$$
T_{\lambda\mu\nu...} = \vartheta^a{}_\lambda\vartheta^b{}_\mu\vartheta^a{}_\nu \ldots T_{abc...} \tag{9}
$$

In the coordinate basis, the components of the spacetime metric are denoted $g_{\mu\nu}$. In the local Lorentz frame, the metric components take the Minkowski form $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$. As in general relativity, the observables are vectors and tensors in the local Lorentz frame. Here we define the observables in Minkowski spacetime as the vierbein decomposition components of the corresponding tensors of physical quantities in de Sitter spacetime. In the present work, we are concerned with the observables of the electromagnetic field, the electric field strength $\mathbf{E}$ and the magnetic field strength $\mathbf{B}$.

First, we introduce the electromagnetic potential contravariant vector

$$
A^\mu = e^\mu_a A^a = \left(\frac{1}{\sqrt{\sigma}}\varphi, \sqrt{\sigma}A_r, \frac{1}{r}A_\theta, \frac{1}{r\sin\theta}A_\phi\right) , \tag{10}
$$

where

$$e^\mu_a = \eta_{ab} g^{\mu\nu} \vartheta^b_\nu = \mathrm{diag}\left(\frac{1}{\sqrt{\sigma}}, \sqrt{\sigma}, \frac{1}{r}, \frac{1}{r\sin\theta}\right),$$
$$A^a = (\varphi, A_r, A_\theta, A_\phi)$$

and A^a are components of the 'ordinary' vector [16] that is that we are seeking for, i.e. the observable vector. Thus the covariant 1-form can be written as below (we define $x^\mu = (t, r, \theta, \phi)$ hereafter),

$$A = A_\mu \mathrm{d}x^\mu = \sqrt{\sigma}\varphi \mathrm{d}t - \frac{1}{\sqrt{\sigma}} A_r \mathrm{d}r - r A_\theta d\theta - r\sin\theta A_\phi \mathrm{d}\phi\,. \tag{11}$$

Then, we can introduce the electromagnetic field strength covariant 2-form $F = \mathrm{d}A = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Accordingly, here F_{ab} are components of the 'ordinary' electromagnetic field strength tensor and they can be written down as a matrix:

$$F_{ab} = \begin{pmatrix} 0 & -E_r & -E_\theta & -E_\phi \\ E_r & 0 & B_\phi & -B_\theta \\ E_\theta & -B_\phi & 0 & B_r \\ E_\phi & B_\theta & -B_r & 0 \end{pmatrix}. \tag{12}$$

Then $F_{\mu\nu}$ becomes

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_r & -r\sqrt{\sigma}E_\theta & -r\sin\theta\sqrt{\sigma}E_\phi \\ E_r & 0 & \frac{r}{\sqrt{\sigma}}B_\phi & -\frac{r\sin\theta}{\sqrt{\sigma}}B_\theta \\ r\sqrt{\sigma}E_\theta & -\frac{r}{\sqrt{\sigma}}B_\phi & 0 & r^2\sin\theta B_r \\ r\sin\theta\sqrt{\sigma}E_\phi & \frac{r\sin\theta}{\sqrt{\sigma}}B_\theta & -r^2\sin\theta B_r & 0 \end{pmatrix}. \tag{13}$$

The action of the electromagnetic field can be written as

$$I_M = \int (-F\wedge *F - A\wedge *j)\,. \tag{14}$$

Here, the symbol '$*$' is the Hodge-dual operator.

To make a comparison with the photon sector of the minimal SME, we can separate the Lagrangian of the electromagnetic field in de Sitter spacetime into one in Minkowski spacetime and a Lorentz violating term as in [7],

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \\ &= -\frac{1}{8}(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}) F^{\kappa\lambda}F^{\mu\nu} - j^\mu A_\mu \\ &= -\frac{1}{8K} R_{\kappa\lambda\mu\nu} F^{\kappa\lambda}F^{\mu\nu} - j^\mu A_\mu \\ &= -\frac{1}{8K}\eta_{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu} - \frac{1}{8K}(R_{\kappa\lambda\mu\nu} - \eta_{\kappa\lambda\mu\nu}) F^{\kappa\lambda}F^{\mu\nu} \\ &\quad - j^\mu A_\mu\,,\end{aligned} \tag{15}$$

where $\eta_{\kappa\lambda\mu\nu} = 2K(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\kappa})$. Comparing with the SME theory, one can obtain an analogous LV coefficient $(k_F)_{\kappa\lambda\mu\nu}$ as [7]:

$$(k_F)_{\kappa\lambda\mu\nu} = \frac{1}{2K}(R_{\kappa\lambda\mu\nu} - \eta_{\kappa\lambda\mu\nu})\,. \tag{16}$$

One useful set is given by [1],

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix} 1+\kappa_{DE} & \kappa_{DB} \\ \kappa_{HE} & 1+\kappa_{HB}\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix},$$

where

$$\begin{aligned}&(\kappa_{DE})^{jk} = -2(k_F)^{0j0k}\,, (\kappa_{HB})^{jk} = \frac{1}{2}\epsilon^{jpq}\epsilon^{krs}(k_F)^{pqrs}\,, \\ &(\kappa_{DB})^{jk} = -(\kappa_{HE})^{jk} = \epsilon^{kpq}\epsilon^{krs}(k_F)^{0jpq}\,.\end{aligned} \tag{17}$$

With some simple calculations, one finds that $(\kappa_{DB})^{jk} = -(\kappa_{HE})^{jk} = 0$. This means that there is no mixing between the electric sector and the magnetic sector in our model. The other two coefficients κ_{DE} and κ_{HB} are not a constant matrix because our model is formulated in curved spacetime and the separation is only valid in the neighborhood of a spacetime point. They appear as:

$$\begin{aligned}\kappa_{HB} &= -\sigma\kappa_{DE} \\ &= K\begin{pmatrix}(x^2)^2+(x^3)^2 & -x^1x^2 & -x^1x^3 \\ -x^1x^2 & (x^1)^2+(x^3)^2 & -x^2x^3 \\ -x^1x^3 & -x^2x^3 & (x^1)^2+(x^2)^2\end{pmatrix},\end{aligned} \tag{18}$$

where $\sigma = 1 - Kr^2$. The most stringent experimental tests on the photon sector of SME comes from the astrophysical test [1]. Following the notation introduced in [1], one can define

$$\begin{aligned}(\tilde{\kappa}_{e+})^{jk} &= \frac{1}{2}(\kappa_{DE}+\kappa_{HB})^{jk}\,, \\ (\tilde{\kappa}_{e-})^{jk} &= \frac{1}{2}(\kappa_{DE}-\kappa_{HB})^{jk} - \frac{1}{3}\delta^{jk}(\kappa_{DE})^{ll}\,, \\ (\tilde{\kappa}_{o+})^{jk} &= \frac{1}{2}(\kappa_{DB}+\kappa_{HE})^{jk}\,, \\ (\tilde{\kappa}_{o-})^{jk} &= \frac{1}{2}(\kappa_{DB}-\kappa_{HE})^{jk}\,, \\ k^a &= \big((k_F)^{0213},\ (k_F)^{0123}, \\ &\quad (k_F)^{0202}-(k_F)^{1313},\ (k_F)^{0303}-(k_F)^{1212}, \\ &\quad (k_F)^{0102}+(k_F)^{1323},\ (k_F)^{0103}-(k_F)^{1223}, \\ &\quad (k_F)^{0203}+(k_F)^{1213},\ (k_F)^{0112}+(k_F)^{0323}, \\ &\quad (k_F)^{0113}-(k_F)^{0223},\ (k_F)^{0212}-(k_F)^{0313}\big)\,,\end{aligned}$$

and

$$(\tilde{\kappa}_{e+})^{jk} = -\begin{pmatrix} -(k^3+k^4) & k^5 & k^6 \\ .[3pt]k^5 & k^3 & k^7 \\ k^6 & k^7 & k^4 \end{pmatrix},$$

$$(\tilde{\kappa}_{o-})^{jk} = \begin{pmatrix} 2k^2 & -k^9 & k^8 \\ -k^9 & -2k^1 & k^{10} \\ k^8 & k^{10} & 2(k^1-k^2) \end{pmatrix}.$$

The most stringent astrophysical tests yield a bound of $|k^a| < 2\times 10^{-32}$ in [1].

In our model $(\tilde{\kappa}_{o-})^{jk} = \frac{1}{2}(\kappa_{DB} - \kappa_{HE})^{jk}$ vanish automatically, while $\tilde{\kappa}_{e+} = \frac{1}{2}(\kappa_{DE} + \kappa_{HB}) = \frac{1}{2}kr^2\kappa_{DE}$, which appears to grow with r and tends to infinity as r approaches the horizon. Since the deviation of de Sitter invariant electrodynamics from Maxwell theory should be minuscule as expected, is this a sign of failure of our model? The answer is no, because we formulate the electromagnetic theory in de Sitter spacetime and the above comparison of our model with minimal SME proceeds and only makes sense in a neighborhood of a spacetime point. The deviation is reasonably large when large scale effects enter. On the other hand, κ_{DE} and κ_{HB} are not a constant matrix, so there is no conventional plane wave solution in our model. The polarization analysis in [1] is invalid here. As a matter of fact, the electrostatic and magnetostatic solutions in our model show that the photon in our model possesses a small mass and it can be estimated that its magnitude is much below the experimental limit obtained in [17].

As with the familiar formulae for gradient, curl, and divergence in the classical curvilinear coordinate systems, we now introduce these things in the spatial part of the local Lorentz frame, or accurately, on the submanifold S^3 of static de Sitter spacetime manifold, by

$$\tilde{\nabla}\psi = \mathrm{d}'\psi = \sqrt{\sigma}\psi_{,\,r}\vartheta^1 + \frac{1}{r}\psi_{,\,\theta}\vartheta^2 + \frac{1}{r\sin\theta}\psi_{,\,\phi}\vartheta^3\,, \tag{19}$$

$$\begin{aligned}\tilde{\nabla}\cdot\mathbf{f} &= *'\mathrm{d}' *' f \\ &= \frac{2}{r}\sqrt{\sigma}f_r + \sqrt{\sigma}f_{r,\,r} + \frac{\cos\theta}{r\sin\theta}f_\theta + \frac{1}{r}f_{\theta,\,\theta} \\ &\quad + \frac{1}{r\sin\theta}f_{\phi,\,\phi}\,,\end{aligned} \tag{20}$$

$$\begin{aligned}\tilde{\nabla}\times\mathbf{f} = *'\mathrm{d}'f &= \left(\frac{\cos\theta}{r\sin\theta}f_\phi + \frac{1}{r}f_{\phi,\,\theta} - \frac{1}{r\sin\theta}f_{\theta,\,\phi}\right)\vartheta^1 \\ &\quad + \left(\frac{1}{r\sin\theta}f_{r,\,\phi} - \sqrt{\sigma}f_{\phi,\,r} - \frac{\sqrt{\sigma}}{r}f_\phi\right)\vartheta^2 \\ &\quad + \left(\frac{\sqrt{\sigma}}{r}f_\theta + \sqrt{\sigma}f_{\theta,\,r} - \frac{1}{r}f_{r,\,\theta}\right)\vartheta^3\,,\end{aligned} \tag{21}$$

$$\begin{aligned}\tilde{\nabla}^2\psi &= (d'\delta' + \delta'd')\,\psi \\ &= \frac{\sqrt{\sigma}}{r^2}\frac{\partial}{\partial r}\left(r^2\sqrt{\sigma}\psi_{,\,r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\psi_{,\,\theta}\right) \\ &\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\,,\end{aligned} \tag{22}$$

where $\mathbf{f}$ is an 'ordinary' 3D vector on S^3 and $\mathbf{f} = f_r\vartheta^1 + f_\theta\vartheta^2 + f_\phi\vartheta^3 = (f_r, f_\theta, f_\phi)$. The vector components on independent basis $\vartheta^i = \vartheta^i_\mu \mathrm{d}x^\mu,\ i = 1,2,3$ point out the orientation of a 3D vector in the local Lorentz frame. We use the symbols tilde and prime to indicate that we do these things on the submanifold S^3.

Noticing that (11)–(13) build a bridge from A^a to $\mathbf{E}$ and $\mathbf{B}$, we can write down these relational equations to show clearly, by performing some elementary calculations, that

$$\mathbf{E} = -\frac{1}{\sqrt{\sigma}}\tilde{\nabla}(\sqrt{\sigma}\varphi) - \frac{1}{\sqrt{\sigma}}\frac{\partial\mathbf{A}}{\partial t} \tag{23}$$

and

$$\mathbf{B} = \tilde{\nabla}\times\mathbf{A}\,, \tag{24}$$

where $\mathbf{A} = (A_r, A_\theta, A_\phi)$, $\mathbf{E} = (E_r, E_\theta, E_\phi)$ and $\mathbf{B} = (B_r, B_\theta, B_\phi)$, respectively.

Noticing that $F = \mathrm{d}A$ is an exact 2-form, we can immediately obtain the Bianchi identity $\mathrm{d}F = \mathrm{d}^2A \equiv 0$ and the dynamical equation $\delta F = *\mathrm{d}*F = j = j_\mu \mathrm{d}x^\mu$, where we define

$$j^\mu = \vartheta^\mu_a j^a = \left(\frac{1}{\sqrt{\sigma}}\rho, \sqrt{\sigma}j_r, \frac{1}{r}j_\theta, \frac{1}{r\sin\theta}j_\phi\right), \tag{25}$$

with $j_a = (\rho, j_r, j_\theta, j_\phi)$. The 'ordinary' electric current density can be defined as before $\mathbf{j} = (j_r, j_\theta, j_\phi)$. Then the electromagnetic field equations in static de Sitter spacetime are easy to obtain as bellow

$$\tilde{\nabla}\cdot\mathbf{B} = 0 \tag{26}$$

$$\tilde{\nabla}\times(\sqrt{\sigma}\mathbf{E}) + \frac{\partial\mathbf{B}}{\partial t} = 0 \tag{27}$$

$$\tilde{\nabla}\cdot\mathbf{E} = \rho \tag{28}$$

$$\tilde{\nabla}\times\mathbf{B} - \frac{1}{\sqrt{\sigma}}\frac{\partial\mathbf{E}}{\partial t} = \mathbf{j}\,. \tag{29}$$

In addition, we study the covariant gauge condition of the electromagnetic field in static de Sitter spacetime. In static de Sitter spacetime, the reasonable gauge condition is the de Sitter covariant gauge condition $\delta A = 0$. In the local Lorentz frame, one can write this equation as follows:

$$\tilde{\nabla}\cdot(\sqrt{\sigma}\mathbf{A}) + \frac{\partial\varphi}{\partial t} = 0\,. \tag{30}$$

This de Sitter gauge will play an important role in dealing with the magnetostatic field.

Now we would like to investigate the interaction between the electromagnetic field and a charged source j^μ in static de Sitter spacetime. As in Lorentz invariant electrodynamics, the Lagrange density of the system reads

$$\mathcal{L}_{\mathcal{M}} = \mathcal{L}_E + \mathcal{L}'_{\mathcal{M}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j_\mu A^\mu \,. \tag{31}$$

Here $\mathcal{L}_E = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ is the purely electromagnetic term and $\mathcal{L}_{\mathcal{M}}$ describes the charged particles (with charge e) and their electromagnetic interactions. Then the electromagnetic force $f^\mu(x)$ can be obtained as

$$f^\mu = F^\mu{}_\gamma j^\gamma = \left(e\frac{\mathbf{v}\cdot\mathbf{E}}{\sqrt{\sigma}}, \sqrt{\sigma} f_r, \frac{1}{r} f_\theta, \frac{1}{r\sin\theta} f_\phi \right) , \tag{32}$$

with

$$\mathbf{f} = (f_r, f_\theta, f_\phi) = e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \,.$$

If $K \to 0$, the LV electrodynamics tends to the Lorentz invariant one and the force returns to the Lorentz force. If we define the purely electromagnetic term of the energy-momentum tensor as usual,

$$T^{\alpha\beta}_{\mathrm{em}} \equiv F^\alpha{}_\gamma F^{\alpha\beta} - \frac{1}{4} g^{\alpha\beta} F_{\lambda\delta}F^{\lambda\delta} \,, \tag{33}$$

we can obtain the energy-momentum conservation law for LV electrodynamics.

$$T^{\alpha\beta}_{\mathrm{em}\,;\,\beta} = -F^\alpha{}_\beta j^\beta = -f^\alpha \,. \tag{34}$$

The semicolon is the abbreviation for the covariant derivative. To make this tensor equation familiar to us and for it to have an obviously observable meaning, we should rewrite the equation using the vierbein formalism as

$$\frac{1}{\sqrt{\sigma}} \tilde{\nabla}\cdot\mathbf{S} + \frac{1}{\sqrt{\sigma}}\frac{\partial\omega}{\partial t} = -\mathbf{j}\cdot\mathbf{E} \,, \tag{35}$$

$$\mathbf{S} = \sigma\,(\mathbf{E}\times\mathbf{B}) \,, \quad \omega = \frac{1}{2}\left(E^2 + B^2\right) ,$$

$$\tilde{\nabla}\cdot \overset{\rightharpoonup\rightharpoonup}{\mathcal{J}} + \frac{1}{\sqrt{\sigma}}\frac{\partial \mathbf{g}}{\partial t} - K\,(\mathbf{r}\times\mathbf{E}\times\mathbf{E}) = -\mathbf{f} \,, \tag{36}$$

and

$$\overset{\rightharpoonup\rightharpoonup}{\mathcal{J}} = -\mathbf{E}\mathbf{E} - \mathbf{B}\mathbf{B} + \frac{1}{2}\overset{\rightharpoonup\rightharpoonup}{\mathcal{I}}\left(E^2 + B^2\right) , \mathbf{g} = \mathbf{E}\times\mathbf{B} \,,$$

where $\mathbf{S}$, ω, $\overset{\rightharpoonup\rightharpoonup}{\mathcal{J}}$, and $\mathbf{g}$ are the energy flux density (Poynting vector), the energy density the electromagnetic stress tensor and the momentum density of the system, respectively. $\overset{\rightharpoonup\rightharpoonup}{\mathcal{I}}$ is the unit tensor in S^3. Equations* (35) and (36) are the vierbein formalism of energy-momentum conservation law of the LV electrodynamics. One can observe again that these equations are different from their cousins in Lorentz invariant formalism.

As an application of the electromagnetic field equations in static de Sitter spacetime, we now introduce the electrostatic field due to a point charged particle. Let us pay some attention to a point charge (with charge q) at the point $\mathbf{r}_0 = (r_0, \theta_0, \phi_0)$. One may ask the question: how can one define a real point charge in the local frame? The answer is connected with the current conservation law in static de Sitter spacetime, that is $\delta j = 0$. Using the vierbein formalism, this conservation law can be written as

$$\tilde{\nabla}\cdot(\sqrt{\sigma}\mathbf{j}) + \frac{\partial\rho}{\partial t} = 0 \,. \tag{37}$$

Rewriting this equation in the form of spherical coordinates, it reads

$$\nabla\cdot\tilde{\mathbf{j}} + \frac{1}{\sqrt{\sigma}}\frac{\partial\rho}{\partial t} = 0 \,, \tag{38}$$

where $\tilde{\mathbf{j}} = (\sqrt{\sigma} j_r, j_\theta, j_\phi)$. The delta function at the point $\mathbf{r_0}$ in S^3 can be written as $\delta'^{\,3}(\mathbf{r}-\mathbf{r_0}) = \sqrt{\sigma}\delta^3(\mathbf{r}-\mathbf{r_0})$, so now we can define the charge distribution function of a point charge as $\rho = q\delta'^{\,3}(\mathbf{r}-\mathbf{r}_0)$. Using the field equations obtained above, the electrostatic field equation becomes

$$-\tilde{\nabla}\cdot\left(\frac{1}{\sqrt{\sigma}}\tilde{\nabla}\left(\sqrt{\sigma}\varphi\right)\right) = q\delta'^{\,3}(\mathbf{r}-\mathbf{r_0}) \,. \tag{39}$$

Utilizing the spherical coordinates, we can transform the equation to appear in the formalism familiar to us

$$-\nabla^2\varphi + K\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + 3K\varphi + 2Kr\frac{\partial\varphi}{\partial r} + \frac{K^2r^2}{\sigma} = q\delta'^{\,3}(\mathbf{r}-\mathbf{r_0}) \,. \tag{40}$$

However, this equation is not easy to solve. Fortunately, there is a way to round this difficulty, because the equation above is de Sitter invariant, so one can perform a suitable ‘quasitranslation’ in static de Sitter spacetime to take the spatial origin $\mathbf{x} = \mathbf{0}$ into any $\mathbf{a}$. Then we can always choose the observed point as the origin of the local frame, namely let $r \to 0$ in (40). We then arrive at

$$-\nabla^2\varphi + 3K\varphi = q\delta^3(\mathbf{r}-\mathbf{r}_0) \,. \tag{41}$$

This equation is very easy to solve by choosing the reasonable boundary condition that $\varphi \to 0$ as $r_0 \to \infty$ (of course, in de Sitter space there is a horizon such that r_0 cannot really go to ∞. However, since the horizon radius R is very large, one can take the horizon as ∞.). We obtain

$$\varphi = \frac{q}{4\pi r_0}\mathrm{e}^{-\sqrt{3K}\,r_0} \,. \tag{42}$$

This electric potential damps a little faster than in Lorentz invariant electrodynamics. The electric field strength $\mathbf{E}$ at the observed point is

$$\mathbf{E} = -q\frac{\mathbf{r}_0}{4\pi r_0^3}\mathrm{e}^{-\sqrt{3K}\,r_0} + q\frac{\sqrt{3K}\mathbf{r}_0}{4\pi r_0^2}\mathrm{e}^{-\sqrt{3K}\,r_0} \,. \tag{43}$$

This formalism is obviously different from the Coulomb theorem. Although the modification is very small, the electrostatic field strength of a point charge in our LV electrodynamics model does not exactly decay as r^{-2}. There is an another exponential damping factor in the potential, which makes the potential looks like a Yukawa one. This effect becomes important in the far field region and it may affect the large-scale universal observation. However, since $K = \frac{1}{R^2} = \frac{A}{3}$ and R could be a very large distance parameter, say the 'radius of universe horizon', the effect of the exponential damping factor can be negligible in the existing experiments.

Next we turn to focus our attention on magnetostatic field in static de Sitter spacetime. The simplest and also the most fundamental case is the magnetic field of a small circle electric current. We set the center of the small circle current (with the electric current strength I and the radius a) at the point $\mathbf{r}_0$, and the observer is at the origin, as in the case of the electrostatic field mentioned above. This case, however, is a little different from the electrostatic field, because the gauge condition we are apt to select is the de Sitter gauge condition (30). Using this gauge condition, one can arrive at

$$\tilde{\nabla} \cdot \mathbf{A} = \frac{K}{\sqrt{\sigma}} \mathbf{r} \cdot \mathbf{A} \,. \tag{44}$$

According to the electromagnetic field equations in static de Sitter spacetime, the differential equation of $\mathbf{A}$ can be written as

$$\tilde{\nabla} \times (\tilde{\nabla} \times \mathbf{A}) = \tilde{\mathbf{j}}(\mathbf{r}) \,, \tag{45}$$

here from (38)

$$\tilde{\mathbf{j}}(\mathbf{r}) = \sqrt{\sigma}\mathbf{j}(\mathbf{r}),\ \mathbf{j}(\mathbf{r}) = j_r\vartheta^1 + j_\theta\vartheta^2 + j_\phi\vartheta^3 \tag{46}$$

is the conserved electric current in the local frame. However, under this formalism, we do not know how to solve it. Thus we should rewrite this equation in spherical coordinates, as we did in the case of the electrostatic field of a point charge. To do this, one should look back at the essential meaning of an 'ordinary' 3D vector in a local frame, actually, a vector in S^3 space $\mathbf{f} = f_r\vartheta^1 + f_\theta\vartheta^2 + f_\phi\vartheta^3$ correspond to a vector in 3D Euclidean space $\mathbf{f} = \frac{1}{\sqrt{\sigma}} f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_\phi\mathbf{e}_\phi$. With this correspondence and defining $\mathbf{A}'(\mathbf{r}) = \frac{1}{\sqrt{\sigma}} A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\phi\mathbf{e}_\phi$, one can obtain the spherical coordinate formalism of the gauge condition (44),

$$\nabla \cdot \mathbf{A}' = 4KrA'_r + Kr^2 A'_{r,\,r} \,. \tag{47}$$

Then the equations of the components of $\mathbf{A}'$ can be derived directly from (45)

$$-(\nabla^2\mathbf{A}')_r + 4KA'_r + 6KrA'_r,\ r + Kr^2 A'_{r,r,r} = \frac{1}{\sqrt{\sigma}} j_r \tag{48}$$

$$-\left(\nabla^2\mathbf{A}'\right)_\theta + \left[K\left(r\left(rA'_\theta\right)_{,r}\right)_{,r} + 3KA'_{r,\theta}\right] = j_\theta \tag{49}$$

$$-(\nabla^2\mathbf{A}')_\phi + \left[K\left(r\left(rA'_\phi\right)_{,r}\right)_{,r} + \frac{3}{\sin\theta}KA'_{r,\phi}\right] = j_\phi \,. \tag{50}$$

Here, however, we have no reason to say that the vector $(\frac{1}{\sqrt{\sigma}} j_r, j_\theta, j_\phi)$ is just a conservation current density in spherical coordinates. Actually, in static de Sitter spacetime, the conservation current density vector must be defined from the (38). Therefore, we can obtain the current strength I through a certain cross section S' in spherical coordinates

$$\begin{aligned} I &= \int (\sqrt{\sigma}\mathbf{j}) \cdot \mathrm{d}\mathbf{S}' \\ &= \int \sqrt{\sigma} j_r r \mathrm{d}\theta' \wedge r' \sin\theta' \mathrm{d}\phi' \\ &\quad + \int j_\theta \mathrm{d}r' \wedge r' \sin\theta' \mathrm{d}\phi' + \int j_\varphi \mathrm{d}r' \wedge r' \mathrm{d}\theta' . \end{aligned} \tag{51}$$

So the conservation current density vector in spherical coordinates is $\mathbf{j}''(r) = \sqrt{\sigma} j_r\mathbf{e}_r + j_\theta\mathbf{e}_\theta + j_\phi\mathbf{e}_\phi$. Under this definition, (48) should be multiplied a factor σ. In the limit of $r \to 0$, one can prove that a symmetric solution of (48)–(50) is a solution of a vector equation as follows. It is easy to show that the solution of (48)–(50) can be obtained by the Bio–Savart theorem for a small circle current placed at the origin in usual Lorentz invariant electrodynamics by setting K to 0. Then pulling the source to $\mathbf{r}_0$, one can obtain A'_r, A'_θ, A'_ϕ with this solution. Supposing that the solution in the spherical frame with origin at the center of the circle is $A'_{r'}, A'_{\theta'}, A'_{\phi'}$, it is easy to show that the only non-vanishing component is $A'_{\phi'}$ and $\mathbf{A}'(0) = A'_{\phi'}\mathbf{e}_{\phi'} = -A'_{\phi'}\sin\theta\sin(\phi-\phi_0)\mathbf{e}_r - A'_{\phi'}\cos\theta\sin(\phi-\phi_0)\mathbf{e}_\theta - A'_{\phi'}\cos(\phi-\phi_0)\mathbf{e}_\phi$, where (r_0, θ_0, ϕ_0) is the spherical coordinates of $\mathbf{r_0}$. Substituting this solution to K terms in (48)–(50) and noting that $\sigma \to 1$ as $r \to 0$, one can obtain the vector equation as follows:

$$-\nabla^2\mathbf{A}'(\mathbf{r}) + 4K\mathbf{A}'(\mathbf{r}) = \mathbf{j}'(\mathbf{r}) \,. \tag{52}$$

This equation is invariant under a translation in parameter $\mathbf{x}$ space, so we can build a spherical coordinate frame (r', θ', ϕ') with origin at the point r_0. It is easy to show that the only non-vanishing component of the solution is still $A'_{\phi'}$ and $\mathbf{A}'(0) = A'_{\phi'}\mathbf{e}_{\phi'} = -A'_{\phi'}\sin\theta\sin(\phi-\phi_0)\mathbf{e}_r - A'_{\phi'}\cos\theta\sin(\phi-\phi_0)\mathbf{e}_\theta - A'_{\phi'}\cos(\phi-\phi_0)\mathbf{e}_\phi$ also holds, where

$$\mathbf{A}' = A'_{\phi'}\mathbf{e}_{\phi'} = \frac{Ia}{4\pi}\oint_0^{2\pi} \frac{\cos\varphi \mathrm{d}\varphi}{\sqrt{r_0^2 + a^2 - 2r_0 a \sin\theta' \cos\varphi}} \times \mathrm{e}^{-\sqrt{4K}\sqrt{r_0^2+a^2-2r_0 a\sin\theta'\cos\varphi}}\mathbf{e}_{\phi'} \,. \tag{53}$$

In the case $2r_0 a \sin\theta' \ll r_0^2 + a^2$, namely in the far field region $(r_0 \gg a)$, and $r_0 \sin\theta' \ll a$, the so-called region of adaxial field, the above integral can be approximately calculated to 3-order

$$A'_{\phi'} = \frac{Ia}{4\pi}\int \mathrm{d}\varphi \cos\varphi \left[\mathcal{P}\frac{r_0 a \sin\theta' \cos\varphi}{(r_0^2+a^2)^{3/2}} + \mathcal{N}\frac{r_0^3 a^3 \sin^3\theta' \cos^3\varphi}{(r_0^2+a^2)^{7/2}}\right] = \frac{Ia}{4\pi}\left[\mathcal{P}\frac{r_0 a \sin\theta'}{(r_0^2+a^2)^{3/2}} + \frac{3}{4}\mathcal{N}\frac{r_0^3 a^3 \sin^3\theta'}{(r_0^2+a^2)^{7/2}}\right], \quad (54)$$

where

$$\mathcal{P} = \frac{\mathrm{e}^{-\sqrt{4K(r_0^2+a^2)}}}{2}\left(1+\sqrt{4K\left(r_0^2+a^2\right)}\right)$$

and

$$\mathcal{N} = \frac{\mathrm{e}^{-\sqrt{4K(r_0^2+a^2)}}}{48}\left\{15+15\sqrt{4K\left(r_0^2+a^2\right)} + 24K\left(r_0^2+a^2\right) + \left[\sqrt{4K\left(r_0^2+a^2\right)}\right]^3\right\}.$$

Pulling the origin back to the field point by setting $\theta' = \pi - \theta_0, \phi' = \phi_0 - \pi$, the magnetic potential $\mathbf{A}$ (at the origin) of the circle electric current can be written as

$$\mathbf{A}(0) = \mathbf{A}'(0) = A'_r\mathbf{e}_r + A'_\theta\mathbf{e}_\theta + A'_\phi\mathbf{e}_\phi = -A'_{\phi'}\sin\theta\sin(\phi-\phi_0)\mathbf{e}_r - A'_{\phi'}\cos\theta\sin(\phi-\phi_0)\mathbf{e}_\theta - A'_{\phi'}\cos(\phi-\phi_0)\mathbf{e}_\phi. \quad (55)$$

This solution shows that the vector potential of the circle current is also a damping potential; it decays a little faster than that in Lorentz invariant electrodynamics as in the case of electrostatic field of a point charge mentioned above. Thus it is reasonable to say that the magnetic field strength also has a damping factor. In the far field region this damping cannot be ignored. In this LV electrodynamics approach, at least on very large-scale observation, the LV effect becomes important and the observation data should be reconsidered because of the damping factor.

We should note that the equations (41) for the scalar potential φ and (52) for vector potential $\mathbf{A}$ are similar to the corresponding equations obtained from Maxwell–Proca equations [17]. The exponential damping factors in the electrostatic and magnetostatic solutions reveal the effect of the effective mass of the photon. However, the effective photon masses for scalar potential φ and vector potential $\mathbf{A}$ are different in our case. With $\frac{m_\gamma c}{\hbar} \sim \sqrt{K} \sim \sqrt{\Lambda}$, one can easily get its approximate magnitude as $m_\gamma \sim 10^{-64}$g, which is far below the photon mass limit $m_\gamma < 1.2\times 10^{-51}$g obtained in [17].

In conclusion, we set up an effective low energy LV classical electrodynamics model in Minkowski spacetime by the covariant formulation of electrodynamics in static de Sitter spacetime. We defined the observable in the model as the vierbein decomposition components of physical tensors. The electromagnetic field equations are obtained in this formalism and their deviation from the Lorentz invariant theory is given. Furthermore, we investigated the energy-momentum conservation law in this LV model. As an application of the LV electromagnetic equations, we studied two basic and simple cases that might be responsible for possible observation confirmation: the electrostatic field of a point charge and the magnetostatic field of a circle electric current. We found that in both cases there is an analogous damping factor in the potential function. This can be regarded as an LV effect and may be important in large-scale observations.

References

1. V.A. Kostelecky, M. Mewes, Phys. Rev. D **66**, 056 005 (2002)
2. R. Jackiw, V.A. Kostelecky, Phys. Rev. Lett. **82**, 3572 (1999)
3. S. Coleman, S.L. Glashow, hep-ph/9812418
4. R. Lehmert, R. Potting, hep-ph/0408285
5. R. Bluhm, V.A. Kostelecky, Phys. Rev. D **71**, 065 008 (2005)
6. V. Alan Kostelecky, Phys. Rev. D **69**, 105 009 (2004)
7. Q.G. Bailey, V.A. Kostelecky, Phys. Rev. D **70**, 076 006 (2004)
8. T. Jacobson, S. Liberati, D. Mattingly, Lorentz violation at high energy: concepts, phenomena and astrophysical constraints, Astro-ph/0505267
9. S. Perlmutter et al., Astrophys. J. **517**, 565 (1999) [astro-ph/9812133]
10. A.G. Riess et al., Astro. J. **116**, 1009 (1998)
11. R.E. Slusher, B.J. Eggleton, Nonlinear Photonic Crystals (Springer, Berlin, 2003), and references therein
12. C.L. Benett et al., Astrophys. J. (Suppl.) **148**, 1 (2003)
13. D. Colladay, V.A. Kostelecky, Phys. Rev. D **58**, 116 002 (1998) [arXiv:hep-ph/9809521]
14. V.A. Kostelecky, Proceedings of the Second Meeting on CPT and Lorentz Symmetry, Bloomington, USA, 15–18 August 2001 (Singapore, World Scientific, 2002)
15. M. Spradlin, A. Strominger, A. Volovich, Les Houches Lectures on de Sitter Space, arXiv:hep-th/0110007
16. S. Weinberg, Gravity and Cosmology: Principles and Applications of the General Theory of Relativity (John Wiley, New York, 1971)
17. J. Luo, L.-C. Tu, Z.-K. Hu, E.-J. Luan, Phys. Rev. Lett. **90**, 081 801 (2003)